[8]Kwakernaak, H., and Sivan, R., *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.
[9]Gajic, Z., and Shen, X., "Decoupling Transformation for Weakly Coupled Linear Systems," *International Journal of Control*, Vol. 50, 1989, pp. 1517–1523.
[10]Su, W.-C., and Gajic, Z., "Decomposition Method for Solving Weakly Coupled Algebraic Riccati Equation," *Journal of Guidance, Control, and Dynamics*, Vol. 15, 1992, pp. 536–538.

Perturbation Analysis for Flexible System Control

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Introduction

FLEXIBLE structures are often described by distributed-parameter models and so are essentially infinite dimensional. For practical considerations, these models are often truncated, and only some of the low-frequency approximated modes are retained for control designs and simulation analyses. However, these modal truncations may lead to control and observation spillover that can destabilize one or more of the poorly damped modes. The problem of spillover suppression to avoid instability has been suggested in the literature using positive real controllers.[1–3] In this Note, the residual mode (pole) shifts that will not be brought into the right-half plane due to structure control interaction is estimated by linking the small perturbation analysis to the robustness tests. It is shown that the derived results are consistent with the positive real control approaches.

Flexible-Structure Control System

Consider flexible structures described by the linear evolution equation

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(0) = x_0 \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

where A is a closed, unbounded, linear differential operator in Hilbert space and

$$Bu(t) = \sum_{i=1}^{m} b_i u_i(t), \qquad y(t) = [y_1(t), \ldots, y_q(t)]^T$$
$$y_i(t) = \langle c_i, x(t) \rangle, \qquad i = 1, \ldots, q$$

with $\langle \cdot, \cdot \rangle$ indicating the inner product on the Hilbert space. For simplicity of exposition, let $\{\lambda_i\}_{i=1}^{\infty}$ be the simple eigenvalues of A, their corresponding normalized eigenvectors denoted by $\{\phi_i\}_{i=1}^{\infty}$. In addition, $\{\psi_i\}_{i=1}^{\infty}$ are the corresponding normalized eigenvectors of A^* (i.e., the self-adjoint operator of A). The biorthogonal pair of eigenvectors are normalized so that

$$\langle \phi_i, \psi_j \rangle = \delta_{ij} \tag{2a}$$

and

$$x = \sum_{i=1}^{\infty} \langle x, \psi_i \rangle \phi_i, \qquad Ax = \sum_{i=1}^{\infty} \lambda_i \langle x, \psi_i \rangle \phi_i \tag{2b}$$

Here, A has only finitely many eigenvalues to the right of $\mathrm{Re}(\lambda) > -\delta$ for $\delta > 0$. We also arrange the simple eigenvalues in an increasing order, $0 < |\lambda_1| < |\lambda_2| < \cdots$. The standard modal expansion procedure can be used to transform model (1) into the decomposed real modal space as

$$\dot{x}_c(t) = A_c x_c(t) + B_c u(t) \qquad x_c(0) = x_{c0} \tag{3a}$$

$$\dot{x}_r(t) = A_r x_r(t) + B_r u(t) \qquad x_r(0) = x_{r0} \tag{3b}$$

$$y(t) = C_c x_c(t) + C_r x_r(t) \tag{3c}$$

where

$$x_c(t) = [\langle x, \psi_1 \rangle, \langle x, \psi_2 \rangle, \ldots, \langle x, \psi_n \rangle]^T$$
$$x_r(t) = [\langle x, \psi_{n+1} \rangle, \ldots]^T$$
$$A_c = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \qquad B_c = [B^*\psi_1, B^*\psi_2, \ldots, B^*\psi_n]^T$$
$$C_c = [C\phi_1, C\phi_2, \ldots, C\phi_n]$$
$$A_r = \mathrm{diag}(\lambda_{n+1}, \ldots), \qquad B_r = [B^*\psi_{n+1}, \ldots]^T$$
$$C_r = [C\phi_{n+1}, \ldots]$$

with $x_c(t)$ and $x_r(t)$ being the controlled and residual states, respectively.

A general observer-based controller designed on the basis of the controlled model is given as

$$\dot{\hat{x}}_c(t) = A_c\hat{x}_c(t) + B_c u(t) + K[y(t) - C_c\hat{x}_c(t)] \quad \hat{x}_c(0) = 0 \tag{4a}$$

$$u(t) = -G\hat{x}_c(t) \tag{4b}$$

where $G \in \mathcal{R}^{m\times n}$ and $K \in \mathcal{R}^{n\times q}$ are, respectively, the control and observer gain matrices. Combining Eqs. (3) and (4), the composite closed-loop system can be expressed in the augmented matrix form

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{e}_c(t) \\ \dot{x}_r(t) \end{bmatrix} = \begin{bmatrix} A_c - B_c G & -B_c G & 0 \\ 0 & A_c - KC_c & KC_r \\ -B_r G & -B_r G & A_r \end{bmatrix} \begin{bmatrix} x_c(t) \\ e_c(t) \\ x_r(t) \end{bmatrix} \tag{5}$$

Stability Criteria

For the actual closed-loop system (5), the following theorem gives a criterion that can be used to justify the closed-loop stability.

Theorem. If G and K are designed so that $A_g = A_c - B_c G$ and $A_e = A_c - KC_c$ are stable matrices and A_r is stable, the closed-loop flexible control system (5) will also be stable if and only if

$$\det\{I_q + \Delta H_r(s)\Pi(s)\} \neq 0 \qquad \forall s \in \mathcal{C}_+ \tag{6}$$

where $\mathcal{C}_+ \equiv \{s \in \mathcal{C}\text{: } \mathrm{Re}(s) \geq 0\}$, $\det(\cdot)$ indicates the matrix determinant, and

$$\Pi(s) = G(sI_n - A_g)^{-1}(sI_n - A_c)(sI_n - A_e)^{-1}K$$

$$\Delta H_r(s) = C_r(sI_\infty - A_r)^{-1}B_r$$

Proof. The closed-loop characteristic equation can be obtained by expanding the following determinant equalities:

$$\det\begin{bmatrix} sI_{2n} - H_{11} & \vdots & -H_{12} \\ \cdots & \cdots & \cdots \\ -H_{21} & \vdots & sI_\infty - A_r \end{bmatrix}$$

$$= \det(sI_{2n} - H_{11}) \det\left[sI_\infty - A_r - H_{21}(sI_{2n} - H_{11})^{-1}H_{12}\right]$$

$$= \det[\Phi(s)] \det\left\{ I_\infty - (sI_\infty - A_r)^{-1}B_r[-G \quad -G] \times (sI_{2n} - H_{11})^{-1}\begin{bmatrix} 0 \\ K \end{bmatrix} C_r \right\}$$

$$= \det[\Phi(s)] \det[I_q + \Delta H_r(s)\Pi(s)]$$

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where

$$H_{11} = \begin{bmatrix} A_c - B_c G & -B_c G \\ 0 & A_c - K C_c \end{bmatrix}, \qquad H_{12} = \begin{bmatrix} 0 \\ K C_r \end{bmatrix}$$

$$H_{21} = [-B_r G \quad -B_r G]$$

and $\det[\Phi(s)] = \det(sI_{2n} - H_{11})\det(sI_\infty - A_r)$. Clearly, the closed-loop stability would be ensured if and only if Eq. (6) is satisfied.

In Eq. (6), $\Delta H_r(s)$ can be viewed as the total spillover that corresponds, in an abstract sense, to control and observation spillover for various time-domain approaches.[4–6] This frequency-dependent perturbation combines the two spillover terms and produces an equivalent destabilizing effect to the nominal design model.

Spillover Effect and Perturbation Analysis

The stability results derived in the literature[4–8] on the basis of unstructured uncertainty assumptions are sufficient but not necessary. Since norm bounds were used, for certain cases, these results tend to be conservative from a practical viewpoint. In the following, the trend on pole shifts of the perturbed eigenvalues will be analyzed without direct application of norm inequalities. The designed pole shifts should be of less concern if $\lambda(A_g)$ and $\lambda(A_e)$ have been designed to achieve large margins of stability. Thus only the trend of the residual pole shifts needs to be discussed.

From Theorem 1, we know that the closed-loop system will be stabilized if and only if the matrix function $I_q + \Delta H_r(s)\Pi(s)$ is invertible for all $s \in \mathcal{C}_+$. Based on this observation, verifying closed-loop stability is equivalent to checking the root distribution of the following characteristic equation:

$$\Phi_{\rm cl}(s) = \det[I_q + \Delta H_r(s)\Pi(s)] = 0 \tag{7}$$

where the total spillover term can be expanded as

$$\Delta H_r(s) = \sum_{i=n+1}^{\infty} \frac{c_{r,i} b_{r,i}^T}{s - \lambda_i}$$

in which $b_{r,i}^T = B^*\psi_i$ and $c_{r,i} = C\phi_i$ are, respectively, the row and the column vectors of B_r and C_r. The polynomial $\phi_{\rm cl}(s)$ can be further expressed in the form

$$\Phi_{\rm cl}(s) = \det\left(I_q + \sum_{i=n+1}^{\infty} \frac{c_{r,i} b_{r,i}^T}{s - \lambda_i}\Pi(s)\right) \tag{8}$$

Now consider the spillover effect on the residual poles λ_i, $i = n+1, n+2, \ldots$. Unless all structural modes are severely overlapped and the perturbation of the high-frequency residual modes resulting from feedback is large, the perturbed eigenvalues $\sigma_k = \lambda_k + \delta\lambda_k$, $k = n+1, n+2, \ldots$, of the residual modes can be assumed to be close to the nominal eigenvalues λ_k, i.e., $\delta\lambda_k$ are small values. Note that a weak coupling condition can be achieved through proper actuator and sensor placement and by shifting eigenvalues of $\lambda(A_g)$ and $\lambda(A_e)$ away from those of the residual operator A_r by means of feedback control. Furthermore $c_{r,i} b_{r,i}^T$ are small for high-order modes. Under this situation, $\Phi_{\rm cl}(\sigma_k)$ with σ_k being the perturbed pole location can be approximated by

$$0 = \Phi_{\rm cl}(\sigma_k) \cong \det\left(I_q + \frac{c_{r,k} b_{r,k}^T}{\sigma_k - \lambda_k}\Pi(\sigma_k)\right) = 1 + \frac{b_{r,k}^T \Pi(\sigma_k) c_{r,k}}{\sigma_k - \lambda_k} \tag{9}$$

This gives

$$\sigma_k \cong \lambda_k - b_{r,k}^T \Pi(\sigma_k) c_{r,k} \tag{10}$$

To obtain a simplified expression that allows quick estimation of σ_k, we take the first-order expansion for $\Pi(\sigma_k)$ in Eq. (10) about λ_k and perform a simple algebraic rearrangement to get

$$\delta\lambda_k \cong -\frac{b_{r,k}^T \Pi(\lambda_k) c_{r,k}}{1 + b_{r,k}^T \Pi'(\lambda_k) c_{r,k}}$$

Since $\Pi(s)$ is strictly proper, we have, at least, $\Pi'(s) = \mathcal{O}(1/s^2)$. Thus, for the high-order residual modes we can obtain an approximate eigenvalue perturbation as

$$\delta\lambda_k \cong -b_{r,k}^T \Pi(\lambda_k) c_{r,k} \tag{11}$$

This expression contains readily available quantities and yields quick estimates of the spillover effect. The perturbed eigenvalue σ_k can now be estimated as

$$\sigma_k \cong \lambda_k - b_{r,k}^T \Pi(\lambda_k) c_{r,k} \tag{12}$$

The term σ_k would be an adequate approximation of the residual poles if structural modes are not strongly overlapped. It is also known that b_k and c_k tend to decrease as k increases for most flexible structures, and the control/observer gains G and K are always designed so that $\Pi(s)$ exhibits a low-pass property [since the poles of $\Pi(s)$ are $\lambda(A_g) \cup \lambda(A_e)$]; also $\Pi(s)$ is strictly proper, i.e., $\Pi(\lambda_k) \to 0$ for $k \to \infty$. These properties together result in $\sigma_k \to \lambda_k$ for $k \to \infty$. That is, the high-order residual modes will not be contaminated by spillover. Thus, concerning the investigation for loop stability, it is worthwhile to analyze Eq. (12) for a few modes that are the nearest ones to the controlled modes because these modes are more sensitive to the spillover effect. In addition, one may use Eq. (12) to justify whether the selected control/observer gains G and K constitute a robust design.

It is also interesting to investigate the residual pole shifts for systems containing collocated actuators and sensors. For this purpose, let the controlled plant (3) be square. Furthermore, we consider the situation where the residual poles lie near the imaginary axis, i.e., $\mathrm{Re}(\lambda_i) \to 0$ for $i = n+1, n+2, \ldots$. Note that this always occurs for a large flexible structure including a very small uniform modal damping or a constant viscous-type damping.

To investigate further, we need the following definitions and related properties for the real positivity of a matrix $H(s)$ of real rational functions of the complex variable $s = \sigma + j\omega$.

Definition.[1,9] An $m \times m$ matrix $H(s)$ of real rational functions is said to be positive real (PR) if following hold:

P1) The matrix $H(s)$ has elements that are analytic for $\mathrm{Re}(s) > 0$.

P2) The poles of $H(s)$ on the $j\omega$ axis are simple and the associated residues are positive semidefinite.

P3) For all $\omega \in \mathcal{R}$ for which $s = j\omega$ is not a pole of any element of $H(s)$, the matrix $H(j\omega) + H^*(j\omega) \geq 0$.

The above properties P2 and P3 can be replaced by the following:

P4) The matrix $H(s) + H^T(s^*) \geq 0\ \forall\, \mathrm{Re}(s) > 0$ with $s = \sigma + j\omega$.

Here, $H(s)$ is termed *strictly positive real* (SPR) if the following hold:

SP1) The matrix $H(s)$ has elements that are analytic for $\mathrm{Re}(s) \geq 0$.

SP2) The matrix $H(j\omega) + H^*(j\omega) > 0\ \forall\, \omega \in \mathcal{R}$.

Lemma.[9,10]

1) A system obtained by the parallel combination of two PR subsystems is also PR.

2) If $H(s)$ is a PR transfer matrix, then $H^{-1}(s)$ is also PR.

Note that in Popov's notation, positive realness is equivalent to hyperstability. Since an SPR matrix is also a PR matrix, results of this lemma can be used for SPR matrices as well.

Since the controlled plant (3) is assumed to be square, i.e., $m = q$, letting $c_{r,k} = b_{r,k}$ in Eq. (12), σ_k now becomes

$$\sigma_k \cong \lambda_k - b_{r,k}^T \Pi(\lambda_k) b_{r,k} \tag{13}$$

That is,

$$\mathrm{Re}(\sigma_k) - \mathrm{Re}(\lambda_k) \cong -\tfrac{1}{2} b_{r,k}^T \left[\Pi(\lambda_k) + \Pi^T(\lambda_k^*)\right] b_{r,k} \tag{14}$$

For the lightly damped flexible structures with $\mathrm{Re}(\lambda_k) \to 0$, $k = n+1, n+2, \ldots$, we may consider the effect of the imaginary part of λ_k alone, and the shift of the real part of σ_k in Eq. (14) will not be positive if

$$\Pi[j\,\mathrm{Im}(\lambda_k)] + \Pi^*[j\,\mathrm{Im}(\lambda_k)] \geq 0 \tag{15}$$

Since $\Pi(s) = \Pi(G, K, s)$, it is obvious from Definition P3 that if the control/observer gains G and K are designed so that $\Pi(s)$ is

PR, then inequality (15) will be satisfied, and the perturbed residual eigenvalues can never be brought into the right-half plane.

For the case where the $\mathrm{Re}(\lambda_k)$ are not sufficiently small but satisfying $\max\{\mathrm{Re}[\lambda(A_g)\cup\lambda(A_e)]\} < \min[\mathrm{Re}(\lambda_k)], k = n+1, n+2, \ldots,$ we can change the complex variable s to be $s = p + \pi$, where $p \in C, \pi = \max\{\mathrm{Re}[\lambda(A_g) \cup \lambda(A_e)]\}$, and express $\Pi(s)$ as

$$\Pi(s) = \Pi(p+\pi) = \tilde{\Pi}(p)$$

Since $\tilde{\Pi}(p)$ is analytic for all $\mathrm{Re}(p) > 0$, using Definition P4, we see that the shift of Eq. (12) will be nonpositive if $\tilde{\Pi}(p)$ is PR, i.e.,

$$\tilde{\Pi}(p) + \tilde{\Pi}^*(p) \geq 0 \qquad \forall \mathrm{Re}(p) > 0$$

For a flexible-structure system described in the form of transfer matrices (i.e., I/O description), it is well known that if a controlled flexible system is PR (respectively, SPR), then an SPR (respectively, PR) compensator will stabilize the closed-loop system despite the existence of spillover.[1] Suppose the controlled flexible system is PR. We now show that an SPR compensator implies the strictly positive realness of $\Pi(s)$. First note that the compensator transfer matrix of Eq. (4) is given by

$$\phi_c(s) = G(sI_n - A_c + B_cG + KC_c)^{-1}K$$

and the controlled dynamics is

$$\phi_p(s) = C_c(sI_n - A_c)^{-1}B_c$$

By the definition of $\Pi(s)$ in Eq. (6), we obtain, successively,

$$\begin{aligned}
\Pi(s) \\
&= G(sI_n - A_c + B_cG)^{-1}(sI_n - A_c)(sI_n - A_c + KC_c)^{-1}K \\
&= G\{sI_n - A_c + B_cG + KC_c + KC_c(sI_n - A_c)^{-1}B_cG\}^{-1}K \\
&= G(sI_n - A_c + B_cG + KC_c)^{-1}\{I_n - K[I_n + C_c(sI_n - A_c)^{-1} \\
&\quad \times B_cG(sI_n - A_c + B_cG + KC_c)^{-1}K]^{-1}C_c(sI_n - A_c)^{-1} \\
&\quad \times B_cG(sI_n - A_c + B_cG + KC_c)^{-1}\}K \\
&= G(sI_n - A_c + B_cG + KC_c)^{-1}K\{I_n - [I_n + C_c(sI_n - A_c)^{-1} \\
&\quad \times B_cG(sI_n - A_c + B_cG + KC_c)^{-1}K]^{-1} \\
&\quad \times C_c(sI_n - A_c)^{-1}B_cG(sI_n - A_c + B_cG + KC_c)^{-1}K\} \\
&= \{[G(sI_n - A_c + B_cG + KC_c)^{-1}K]^{-1} \\
&\quad + C_c(sI_n - A_c)^{-1}B_c\}^{-1} \qquad (16)
\end{aligned}$$

The matrix inversion formula has been used in the third step above. From Lemma 1, we know that an SPR transfer matrix $\phi_c(s)$ implies the strictly positive realness of $\phi_c^{-1}(s)$. Since $\phi_p(s)$ is assumed to be PR, based on the basic properties of the parallel combination of the PR blocks $\phi_c^{-1}(s)$ and $\phi_p(s)$, we may then conclude from Lemma 1 and Eq. (16) that $\Pi(s)$ is SPR. This implies the shifts of $\mathrm{Re}(\sigma_k)$, $k = n+1, n+2, \ldots,$ in Eq. (14) would not be positive. Also note that if $\phi_p(s)$ is SPR, then a PR transfer matrix $\phi_c(s)$ implies the strictly positive realness of $\Pi(s)$ and the shifts of residual poles could still be nonpositive.

Conclusions

In this Note, we have investigated residual pole shifts of a class of flexible structures under low-order observer-based control. Permissible magnitudes of residual pole shifts that ensure residual poles retained in the open left-half plant given that the closed-loop is stable have been presented. It was shown that the derived results are consistent with the positive real control approaches.

References

[1]Benhabib, R. J., Iwens, R. P., and Jacson, R. L., "Stability of Large Space Structure Control Systems using Positivity Concepts," *Journal of Guidance and Control*, Vol. 4, No. 5, 1981, pp. 487–494.

[2]Joshi, S. M., *Control of Large Flexible Space Structures*, Springer–Verlag, Berlin, 1989.

[3]McLaren, M., and Slater, G. L., "Robust Multivariable Control of Large Space Structures Using Positivity," *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 4, 1987, pp. 393–400.

[4]Balas, M. J., "Feedback Control of Flexible Systems," *IEEE Transactions on Automatic Control*, Vol. AC-23, 1978, pp. 673–679.

[5]Skelton, R. E., *Dynamic Systems Control*, Wiley, New York, 1988.

[6]Tang, T. S., and Hung, G., "On Control of Large Space Structures," presented at 1990 American Control Conference, San Diego, CA, May 1990, pp. 2965–2970.

[7]Lin, C. L., "Robust Control of Flexible Structures Using Residual Mode Filters," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 5, 1993, pp. 973–977.

[8]Vidyasagar, M., *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge, MA, 1985.

[9]Landau, Y. D., *Adaptive Control: The Model Reference Approach*, Dekker, New York, 1979.

[10]Popov, V. M., *Hyperstability of Control Systems*, Springer, New York, 1966.

Regulation of Single-Link Flexible Manipulator Involving Large Elastic Deflections

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I. Introduction

THE control of single-link flexible manipulators has been widely studied in the literature. A vast majority of these studies (too numerious to mention) have designed controllers based on truncated finite-dimensional models. Only in a few related papers[1–7] has there been construction of controllers making direct use of distributed parameter models. However, the small-deflection assumption has been invoked in nearly all these studies (using either truncated finite-dimensional models or infinite-dimensional models).

In this Note, the rest-to-rest maneuver of a horizontally slew torque-driven beam undergoing geometrically exact elastic deflections is considered. The equations of motion for a hub–beam system are derived first. Instead of the approach[8,9] employing Hamilton's principle, this derivation uses Newton's second law for the sake of retaining the simple physical structure of the problem. Then a simple linear feedback law using only the joint torque is obtained via a Lyapunov-type method. In deriving the feedback law, neither model truncation nor small deflection assumption is imposed. A proof of globally asymptotic stability of the closed-loop system is presented. A similar problem was considered in Ref. 9, but no formal proof of asymptotic stability was provided.

We emphasize that 1) to admit large elastic deflections removes the speed and acceleration limitations imposed by the small-deflection assumption, 2) to adopt the distributed parameter modeling eliminates the truncation error caused by model simplification, and 3) to employ the geometrically exact description implies that no error in kinematics of elastic deformation is involved.

II. Equations of Motion

The horizontal slewing beam in the undeformed state depicted in Fig. 1 of length L, area moment of inertia I, cross-sectional area

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